Examples

$e^{w} = \sum_{n=0}^{\infty} \frac{w^n}{n!}$

1) Find the Taylor series for $f(z) = e^{z^2}$ at $z_0 = 0$, and its radius of convergence.

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

way better than using $\frac{f^{(n)}(z_0)}{n!}$!!

subs.

$(R = \infty)$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1. \qquad 1 + z +$$

2) Find the Taylor series for $f(z) = \dfrac{1}{(z-1)^2}$ at $z_0 = 0$, along with its radius of convergence.

$$\frac{d}{dz}(1-z)^{-1} = -(1-z)^{-2}(-1) = \frac{1}{(z-1)^2}.$$

diff

$$\frac{1}{(z-1)^2} = \sum_{n=1}^{\infty} n z^{n-1} \left(= \sum_{k=0}^{\infty} (k+1) z^k \right) \qquad |z| < 1$$

$k = n-1$

$k+1 = n$

3) Find the Taylor series for $f(z) = \log(1+z)$ at $z_0 = 0$, along with its radius of convergence.

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n$$

geometric

$|-z| < 1$

$|z| < 1$

$R = 1.$

integrate!

$$\log(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} + C \quad \text{to be continued} \ldots$$

$$\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} + \cancel{\log(1+0)}^{\,0}$$

$k = n+1$

$R = 1.$

(also easy as a Taylor series)

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad |w|<1$$

4) Find the Taylor series of $f(z) = \dfrac{1}{z^2 - z - 6} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

$$\frac{1}{z-3} = \frac{1}{(z-z_0)-(3-z_0)}$$

$$f(z) = \frac{1}{5}\,\frac{1}{-3}\left(\frac{1}{1-\frac{1}{3}z}\right) - \frac{1}{5}\,\frac{1}{2}\,\frac{1}{1+\frac{1}{2}z}$$

$$1+\tfrac{1}{2}z = 1-\left(-\tfrac{1}{2}z\right)$$

$$= -\frac{1}{15}\left(\sum_{n=0}^{\infty}\frac{1}{3^n}z^n\right) - \frac{1}{10}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}z^n\right)$$

$\left|\frac{z}{3}\right|<1$ $\quad R=3$

$\left|\frac{z}{2}\right|<1$ $\quad R=2$

$R=2$

you could combine into a single series, but is actually better

5) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1+i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

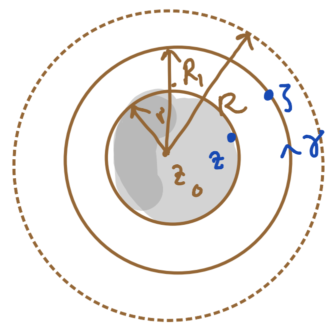
later

Theorem 4 If f is analytic in $D(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $D(z_0; R)$. Thus the radius of convergence of the power series is at least R.

proof: Let $|z - z_0| \le r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{it}$, $0 \le t \le 2\pi$, the circle $\zeta - z_0 = R_1$.



Then the Cauchy integral formula reads

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We use geometric series magic:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$\zeta - z = (\zeta - z_0) - (z - z_0)$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \dfrac{(z - z_0)}{(\zeta - z_0)}}\, d\zeta$$

factor out $\zeta - z_0$

$\left|\frac{z - z_0}{\zeta - z_0}\right| \le \frac{r}{R} = \mu < 1.$

using the geometric series for $\dfrac{1}{1 - w}$ with $|w| \le \dfrac{r}{R_1}$:

$\forall \zeta$ s.t. $|\zeta - z_0| = R_1$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{1}} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

$$= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} f(\zeta) \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z - z_0|^n}{|\zeta - z_0|^{n+1}} \le \frac{1}{R_1}\left(\frac{r}{R_1}\right)^n,$$

the series which is the integrand converges absolutely uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z - z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \cancel{\sum_{n=0}^{\infty}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\gamma} \cancel{\sum_{n=0}^{\infty}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the Cauchy integral formula for derivatives!!!

Q.E.D.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Math 4200
Wednesday October 28
3.2 Finish Taylor series/power series facts and examples. After we quickly review the first page of today's notes we'll go back and finish the examples from Monday, as well as the proof of the theorem that connects radius of convergence for Taylor series of analytic functions at z_0 to the maximal disk of analyticity.

sometimes a shortcut

Announcements: • Quiz today!

• if you want to do a project in lieu of final exam, let me know (& discuss) by next Friday.

Warmup-exercise: What is the power series for $\cos(z)$ based at the origin, and what is its radius of convergence? 'til 11:53

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \qquad R = \infty$$

two ways:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\Big[1 + \cancel{iz} - \frac{z^2}{2!} - \cancel{\frac{iz^3}{3!}} + \frac{z^4}{4!} + \dots$$

$$+ 1 - \cancel{iz} - \frac{z^2}{2!} + \cancel{\frac{iz^3}{3!}} + \frac{z^4}{4!} \dots$$

✓

Taylor series

$$\cos z = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{f^{(n)}(0)}{n!} \qquad \cos^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ 1 & n = 4k \\ -1 & n = 4k+2 \end{cases}$$

3.2 Quick fact Summary from Monday:

- Theorem 1 Every *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C}.$$

has a unique radius of convergence $R \in [0, \infty]$ such that the power series above converges $\forall\, z$ with $|z - z_0| < R$ (uniformly absolutely for any $r < R$), and diverges for all z with $|z - z_0| > R$. The limit is an analytic function.

- Theorem 2 Power series may be differentiated and integrated term by term to get derivatives and antiderivatives of f, and the resulting power series have the same radius of convergence as the original function f.

- Theorem 3 Therefore, whenever $R > 0$, after differentiating k times and substituting $z = z_0$ into the power series, one realizes that the power series is actually the Taylor series for f,

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

As a consequence, power series that yield the same analytic function in a neighborhood of z_0 must be identical, because they are the Taylor series for that function, centered at z_0.

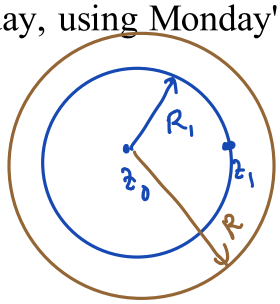
Theorem 4 f f is analytic in $D(z_0; R_1)$ then the Taylor series for f at z_0,

to be proven today!

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to f in $D(z_0; R_1)$. Thus the radius of convergence of the Taylor series is at least R_1. And, one can use this to get an upper bound on the radius of convergence: if $\exists\, z_1$ such that f cannot be extended to be analytic at z_1, then the radius of convergence of the Taylor series is at most $|z_1 - z_0|$, since a larger radius of convergence would imply that a possible domain of analyticity contains z_1.

(To be proved today, using Monday's notes.)



Sometimes it is useful to know you can multiply power series term by term, and without having to worry about radius of convergence issues. This theorem makes it a breeze:

<u>Theorem 5</u> (Multiplying power series): Let

$$\begin{cases} f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \left(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\right) \\ g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n = \left(b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \ldots\right) \end{cases}$$

in $D(z_0; R)$. Then the power series for $f(z)g(z)$ also converges in $D(z_0; R)$ and is given by

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(z-z_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z-z_0)^2 + \ldots.$$

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) (z-z_0)^n,$$

in other words, what you expect by formally multiplying and collecting all coefficients for each $(z-z_0)^n$.

proof: We know that power series are Taylor series. Therefore,

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(z_0)}{n!} (z-z_0)^n$$

will converge in $D(z_0; R)$. Compute the various derivatives, using the product rule for first, second, ..., n^{th} derivatives of product functions (via induction and the binomial theorem).

$$(fg)(z_0) = a_0 b_0$$
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) = a_1 b_0 + a_0 b_1$$
$$(fg)''(z_0) = f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0)$$
$$(fg)''(z_0) = (2a_2)b_0 + 2a_1 b_1 + a_0(2b_2) = 2!(a_2 b_0 + a_1 b_1 + a_0 b_2)$$

In general and using the product rule, (checked by induction, as in proof of binomial theorem in first HW),

$$(fg)^{(n)}(z_0) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(z_0) g^{(n-j)}(z_0)$$

$$= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (j!\, a_j)\ (n-j)!\, b_{n-j} = n! \sum_{j=0}^{n} a_j b_{n-j}$$

Q.E.D.